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PROJECTING PRECIPITOUSNESS

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ABSTRACT

This paper is about "strong" ideals on small cardinals. It is shown that a typical property of large cardinal measures does not transfer to these ideals. More specifically, that precipitous ideals on $\mathcal{P}_{\omega_1}\lambda$ spaces may not project down to precipitous ideals on "smaller" $\mathcal{P}_{\omega_1}\lambda'$ spaces. Also, that the existence of a presaturated ideal on the bigger space does not imply the existence of a presaturated ideal on the smaller space.

1. Introduction

Precipitous ideals on $\mathcal{P}_\kappa\lambda$ resemble in many ways supercompact ultrafilters (for κ, λ uncountable cardinals, κ regular). In the supercompact case, we have that if κ is supercompact and $\kappa \le \lambda' < \lambda$ (λ', λ cardinals), then the supercompact ultrafilter on $\mathcal{P}_\kappa\lambda$ *induces* a supercompact ultrafilter on $\mathcal{P}_\kappa\lambda'$: The *projection* of a fine, normal ultrafilter on $\mathcal{P}_\kappa\lambda$ onto $\mathcal{P}_\kappa\lambda'$ is itself a fine, normal ultrafilter (the notion of *projection* will be defined below). In the case of ideals, we have that the projection of a normal ideal on $\mathcal{P}_\kappa\lambda$ is a normal ideal on $\mathcal{P}_\kappa\lambda'$ ($\kappa \le \lambda < \lambda'$ uncountable cardinals, κ regular). It is natural then to ask whether the projection of a precipitous ideal on $\mathcal{P}_\kappa\lambda$ onto $\mathcal{P}_\kappa\lambda'$ need itself be precipitous.

In this paper we show that the answer is negative. It is negative for all regular, uncountable, cardinals κ. However, our main focus is on $\kappa = \omega_1$. We show that, assuming large cardinals, for any regular uncountable λ', λ with $\lambda' < \lambda$ (assuming $\lambda'^{<\lambda'} < \lambda$) there is a model where the NS (non-stationary) ideal on $\mathcal{P}_{\omega_1}\lambda$ is precipitous, while its projection, the NS ideal on $\mathcal{P}_{\omega_1}\lambda'$, is non-precipitous. In fact, in our model the NS ideal on $\mathcal{P}_{\omega_1}\lambda$ is presaturated, thus giving an example of a presaturated ideal which does not project down to a precipitous ideal.

Given that the projection of a precipitous ideal on $\mathcal{P}_{\omega_1}\lambda$ to $\mathcal{P}_{\omega_1}\lambda'$ need not itself

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be precipitous, one may still ask: does the existence of a precipitous ideal on $\mathcal{P}_{\omega_1}\lambda$ imply the existence of *any* precipitous ideal on $\mathcal{P}_{\omega_1}\lambda'$? We do not know the answer to this question, but we answer a related question: We show that the existence of a presaturated ideal on $\mathcal{P}_{\omega_1}\lambda$ does not imply the existence of a presaturated ideal on $\mathcal{P}_{\omega_1}\lambda'$.

The construction of our models proceeds in two stages: Given regular uncountable $\lambda' < \lambda$, we first construct a model where there are no presaturated ideals on $\mathcal{P}_{\omega_1}\lambda'$ or where the NS ideal on $\mathcal{P}_{\omega_1}\lambda'$ is not precipitous. We then force over either one of these models to get a presaturated ideal on $\mathcal{P}_{\omega_1}\lambda$ without effecting the facts established in the first stage.

We note that Laver has an example in [L] of a precipitous ideal on $\mathcal{P}_{\omega_1}\lambda$ which projects down to a non-precipitous ideal on ω_1. His ideal, the so-called "master-condition" ideal, is not the NS ideal and can be defined only in certain forcing constructions.

Our notation is standard. We use $\mathcal{P}_\kappa\lambda$ for the set of all subsets of λ of cardinality less than κ, and, in general, $\mathcal{P}_\kappa X$ for subsets of X of cardinality less than κ. An ideal on $\mathcal{P}_\kappa\lambda$ is a subset of $\mathcal{P}(\mathcal{P}_\kappa\lambda)$; we will only consider κ-complete and fine ideals. We will discuss the NS ideal on $\mathcal{P}_{\omega_1}\lambda$. A subset of $\mathcal{P}_\kappa\lambda$ is NS iff it is disjoint from a club subset of $\mathcal{P}_\kappa\lambda$. In general, there are two distinct notions of "club" subsets of $\mathcal{P}_\kappa\lambda$, club and strongly club. (A set $C \subset \mathcal{P}_\kappa\lambda$ is club in $\mathcal{P}_\kappa\lambda$ iff it is unbounded in $\mathcal{P}_\kappa\lambda$, i.e., every set in $\mathcal{P}_\kappa\lambda$ is covered by a set in C, and it is closed, i.e., the union of any increasing sequence of length less than κ of sets in C is itself in C; C is strongly club in $\mathcal{P}_\kappa\lambda$ iff there is a structure $\mathfrak{A}$ of the form $\mathfrak{A} = \langle \lambda, f_i \rangle_{i\in\omega}$, where $f_i : \lambda^{<\omega} \to \lambda$ and $C = \{N \prec \mathfrak{A} \mid |N| < \kappa\}$.) For a discussion of the two notions, see [FMS]. However, in the case we are interested in, i.e., $\kappa = \omega_1$, the two notions coincide (see [Kue]). The notions of club and NS for subsets of $\mathcal{P}_\kappa X$ are defined in a similar way. We will discuss two types of ideals: precipitous and presaturated. An ideal on $\mathcal{P}_{\omega_1}\lambda$ is precipitous iff the generic ultrapower associated with it is well-founded. An ideal on $\mathcal{P}_{\omega_1}\lambda$ is presaturated iff forcing over the positive sets associated with it does not collapse $(\lambda)^+$. (The former were introduced by Jech and Prikry in [JP], the latter by Baumgartner and Taylor in [BT].)

2. In this section we show how to construct models of set-theory, using small forcing, with no presaturated ideals on a given space X. We will consider two such cases: The first, when X is some successor cardinal κ and the second, when X is $\mathcal{P}_\kappa\lambda$ for some λ. We note that we are attempting to construct the models with *small* forcing, so that large cardinals (larger than the size of the forcing) will be preserved in the forcing extension.

We now discuss the forcing for "destroying" all presaturated ideals on a regular successor κ. What the forcing does is add a function $f: \kappa \to \kappa$ which is guaranteed to represent an ordinal $\geq (\kappa^+)^V$ in *any* generic ultrapower associated with *any* ideal on κ. For an ideal I on κ, let j_I be the associated elementary embedding. Since $j_I(\kappa) > [g]$ for any $g: \kappa \to \kappa$, the presence of a function f as above implies that $j_I(\kappa) > (\kappa^+)^V$. For κ a successor cardinal, this, in turn, implies that $(\kappa^+)^V$ is collapsed in the generic ultrapower and thus I could not have been presaturated.

The idea is to use functions $\langle O_\beta \mid \kappa \leq \beta < \kappa^+ \rangle$, $O_\beta: \kappa \to \kappa$, which can be shown to represent the ordinals greater than or equal to κ and smaller than κ^+ in any generic ultrapower generated by a normal ideal on κ^+. We add a function $f, f: \kappa \to \kappa$ and make sure that f eventually dominates each of these functions (i.e., $\forall \alpha < \kappa^+$ $\exists \beta (\kappa < \beta < \kappa^+$ and $(\gamma > \beta \Rightarrow f(\gamma) > O_\alpha(\gamma))))$ and thus f has to represent an ordinal greater than those represented by the O_β functions, i.e., it has to represent *at least* κ^+. (If the ideal is not normal, O_β may not represent β, but it will represent an ordinal $\geq \beta$; thus we will still get that f represents an ordinal $\geq \kappa^+$.)

We will now describe the O_β functions and then we will describe the forcing we use to destroy all presaturated ideals. Both the O_β functions and the forcing notion used will also be relevant to the construction of a model where the NS ideal either on κ or on $\mathcal{P}_\kappa \lambda$ is not precipitous.

We note that even though we will prove the next theorem only for successor κ's, the discussion of the O_β functions and the definition of the forcing notion we will use, as well as the basic combinatoric properties of this forcing notion, hold for any regular κ. We will use this later on. The O_β functions defined below are often referred to in the literature as the "canonical functions on κ".

O_β functions, $\kappa \leq \beta < \kappa^+$

For each $\kappa \leq \beta < \kappa^+$, fix a well-ordering W_β of κ of order type β. For $\alpha < \kappa$ define $O_\beta(\alpha)$ to be the order type of $(W_\beta \restriction \alpha \times \alpha)$. Then $O_\beta: \kappa \to \kappa$.

Let I be any normal ideal on κ, let G be generic over I^+ and let $j: V \to M$ be the associated generic elementary embedding. We then have that

$$j(O_\beta)(\kappa) = \mathrm{ot}(j(W_\beta) \restriction \kappa \times \kappa) = \mathrm{ot}(W_\beta) = \beta,$$

since κ is the critical point of j (we use here $\mathrm{ot}(x)$ for the order type of x).

Thus, since I was a normal ideal, O_β represents β in the generic ultrapower. (If

I was not normal, we would still have $[O_\beta] = j(O_\beta)[id] = \mathrm{ot}(j(W_\beta) \restriction \alpha \times \alpha)$, for some $\alpha > \kappa$, where *id* stands for the identity function on κ. Thus $[O_\beta] \geq \beta$.)

This information about the O_β functions is sufficient for constructing a "small" forcing notion for destroying all presaturated ideals on κ. However, for getting a model where the NS ideal on either κ or $\mathcal{P}_\kappa\lambda$ is not precipitous, we will need a further property of the O_β functions. We mention this property here:

Let $\kappa \leq \alpha < \beta < \kappa^+$. Then there is a club $C \subset \kappa$ such that $\gamma \in C \Rightarrow O_\alpha(\gamma) < O_\beta(\gamma)$.

Proof. Let O_α be defined with respect to W_α. Let O_β be defined with respect to W_β. Let δ be the α'th ordinal in W_β and let $g : \kappa \to \kappa$ be an isomorphism between W_α and the initial segment of W_β of order type α.

Let $C = \{\xi < \kappa \mid \xi > \delta \wedge (\sigma < \xi \Rightarrow g(\sigma) < \xi)\}$. C is club in κ and $C \subset \{\gamma < \kappa \mid O_\alpha(\gamma) < O_\beta(\gamma)\}$ since $\xi \in C \Rightarrow (\forall \alpha < \xi\ g(\alpha) < \xi) \Rightarrow (\mathrm{ot}(W_\alpha \restriction \xi \times \xi) \leq \mathrm{ot}(W_\beta \restriction \xi \times \xi))$; but $\mathrm{ot}(W_\beta \restriction \xi \times \xi)$ has δ "after" $g''W_\alpha \restriction \xi \times \xi$; thus $\forall \xi \in C$, $\mathrm{ot}(W_\alpha \restriction \xi \times \xi) < \mathrm{ot}(W_\beta \restriction \xi \times \xi)$, so that $\forall \xi \in C\ O_\alpha(\xi) < O_\beta(\xi)$.

Now we go back to destroying all presaturated ideals on κ: In order to do that, it is sufficient to add a function $f : \kappa \to \kappa$ which eventually dominates all O_β functions.

Consider the following partial order P^1_κ. Each $p \in P^1_\kappa$ has two parts, $p = \langle p_1, p_2 \rangle$. The first, p_1, is a partial function from κ to κ of size less than κ (p_1 is an approximation of f). The second, p_2, is a set of size less than κ of ordinals greater than κ and smaller than κ^+ (p_2 is the set that tells us which O_β functions have to be dominated in any extension of p). Regarding the order on P^1_κ: If $p, q \in P^1_\kappa$, $p = \langle p_1, p_2 \rangle$, $q = \langle q_1, q_2 \rangle$ then $q \leq p$ (q stronger than p) iff $q_1 \supset p_1$, $q_2 \supset p_2$ and if $\langle \alpha, \beta \rangle \in q_1 \backslash p_1$ then $\beta > O_\gamma(\alpha)$, for all γ in p_2.

We are now ready to state the theorem we want. This theorem was pointed out to me by Magidor:

Theorem 2.1. *Let κ be an uncountable successor cardinal. Assume $V \models \kappa^{<\kappa} = \kappa$. Let P^1_κ be as defined above, and let G be a generic filter over P^1_κ. Then in $V[G]$, there are no presaturated ideals on κ. (Thus there is a small forcing which destroys all presaturated ideals on κ.)*

Proof. By density arguments, it is clear that if $f = \bigcup\{p_1 \mid \langle p_1, p_2 \rangle \in G\}$, then $f : \kappa \to \kappa$ and f eventually dominates each O_β function. Thus, by our previous arguments, there will be no presaturated ideals on κ in $V[G]$, provided no cardinals are collapsed.

Now P_κ^1 is $<\kappa$-closed, so no cardinals less than or equal to κ are collapsed. We will show that P_κ^1 has the κ^+-c.c. and thus κ^+ and all greater cardinals are not collapsed. This will complete the proof.

CLAIM 2.2. *P_κ^1 has the κ^+-c.c.*

PROOF. When referring to $p, q \in P_\kappa^1$ we will always assume $p = \langle p_1, p_2\rangle$, $q = \langle q_1, q_2\rangle$.

Let A be an antichain of P_κ^1 of size κ^+. For each $p \in P_\kappa^1$, p_1 is a set of size less than κ of pairs of ordinals, each less than κ. Therefore (since $\kappa^{<\kappa} = \kappa$), there are only κ many possibilities for p_1. Hence $\exists B \subset A$, $|B| = \kappa^+$ such that $p, q \in B \Rightarrow p_1 = q_1$. But then each two members p, q of B must be compatible since $\langle p_1 = q_1, p_2 \cup q_2\rangle$ is such a common extension of p, q. This is a contradiction. ∎

The proof of the claim completes the proof of the theorem. ∎

We now state the analogue of this theorem for the $\mathcal{P}_\kappa\lambda$ case. We will only apply it in the case $\kappa = \omega_1$, but as the general proof (for κ a successor cardinal) is no more involved than that for the case $\kappa = \omega_1$, we present it here.

THEOREM 2.3. *Let $\kappa < \lambda$ be regular uncountable cardinals, κ a successor cardinal. Assume $V \models \lambda^{<\lambda} = \lambda$ and $|(\lambda^+)^{<\lambda}| = \lambda^+$. Then there is a notion of forcing, $P_{\kappa,\lambda}^2, |P_{\kappa,\lambda}^2| = \lambda^+$, such that if G is generic over $P_{\kappa,\lambda}^2$ then, in $V[G]$, there are no presaturated ideals over $\mathcal{P}_\kappa\lambda$. Further, no cardinals are collapsed in $V[G]$.*

PROOF. The proof follows the same tack as before. We first need analogues of the O_β functions.

Let $\langle W_\beta \mid \lambda \leq \beta < \lambda^+\rangle$ be a sequence of well orderings of λ, where W_β is a well ordering of λ of order type β. Define $\langle O'_\beta \mid \kappa \leq \beta < \kappa^+\rangle$, $O'_\beta : \mathcal{P}_\kappa\lambda \to \kappa$ as follows:

$$O'_\beta(X) = \mathrm{ot}(W_\beta \restriction X \times X), \qquad X \in \mathcal{P}_\kappa\lambda.$$

As before, O'_β represents β in any generic ultrapower coming from a normal ideal on $\mathcal{P}_\kappa\lambda$ $(j(O'_\beta)(j''\lambda) = \mathrm{ot}(j(W_\beta) \restriction j''\lambda \times j''\lambda) = \mathrm{ot}(W_\beta) = \beta)$ and an ordinal $\geq \beta$, for any ideal on $\mathcal{P}_\kappa\lambda$.

Next we define the partial order $P_{\kappa,\lambda}^2$, the analogue of P_κ^1. Again, members of $P_{\kappa,\lambda}^2$ are pairs $p = \langle p_1, p_2\rangle$, where p_1 is a partial function from $\mathcal{P}_\kappa\lambda$ to κ of size $< \lambda$ and p_2 is a set of size less than λ of ordinals greater than or equal to λ and smaller than λ^+. If $p = \langle p_1, p_2\rangle$, $q = \langle q_1, q_2\rangle$ then $q \geq p$ iff $q_1 \supset p_1$, $q_2 \supset p_2$ and if $\langle X, \beta\rangle \in q_1 \setminus p_1$ then $\beta > O'_\gamma(X)$ for all γ in p_2.

Forcing with $P_{\kappa,\lambda}^2$ does not collapse cardinals: The forcing is $< \lambda$ closed and has the λ^+ c.c. The argument for the λ^+ c.c. is the same as before (for P^1_κ), using $\lambda^{<\lambda} = \lambda$, and we omit it here.

Finally, if $f = \bigcup\{p_1 \mid \langle p_1, p_2\rangle \in G\}$ then $f : \mathcal{P}_\kappa\lambda \to \kappa$ and f eventually dominates each O'_β function, i.e., for each $\beta, \lambda \leq \beta < \lambda^+$, there is a club set $C \subset \mathcal{P}_\kappa\lambda$ of the form $C_\beta = \{X \in \mathcal{P}_\kappa\lambda \mid \gamma_\beta \in X\}$ (where $\gamma_\beta \in \lambda$) such that $\forall X \in C_\beta f(X) > O'_\beta(X)$. Thus the ordinal f represented in a generic ultrapower by a fine ideal will be greater than those represented by the O'_β functions, i.e., greater than or equal to λ^+. Hence $j(\kappa) > \lambda^+$ for any elementary embedding j arising out of such an ultrapower, and thus any fine ideal on $\mathcal{P}_\kappa\lambda$ will be not presaturated. ∎

3. We now discuss the construction of models of set theory, using small forcing, in which the NS ideal on $\mathcal{P}_\kappa\lambda$ ($\kappa < \lambda$ regular cardinals) is not precipitous. (We do not know how to get such models where *all* ideals are not precipitous.) Our construction is a generalization of the construction of models for set theory where the NS ideal on κ is non-precipitous. We first present a proof for this case, and only then a proof for the $\mathcal{P}_\kappa\lambda$ case. For both cases we will use an equivalent characterization of precipitousness, different from the one mentioned above. This characterization was first given in [GJM].

Let $\kappa \leq \lambda$ be uncountable cardinals, κ regular. Let I be an ideal on $\mathcal{P}_\kappa\lambda$. Consider the following game, which we will call $G(I)$:

Players I and II both play sets in I^+. Player I plays, in addition, functions from subsets of $\mathcal{P}_\kappa\lambda$ to ORD. The game proceeds as follows:

$$\begin{array}{llllll} \text{I} & A_1, f_1 & & A_2, f_2 & & \cdots \\ \text{II} & & B_1 & & B_2 & \cdots \end{array}$$

where $A_1 \supset B_1 \supset A_2 \supset B_2 \cdots$.

Player I wants $\bigcap_{i\in\omega} A_i = \emptyset$, and the functions he plays are required to witness this: He must play $f_i : A_i \to \mathrm{ORD}$ and $\forall X \in A_{i+1}\, f_i(X) > f_{i+1}(X)\,(i \in \omega)$. Player I wins iff the game does not stop.

Lemma 3.1. *Let $\kappa < \lambda$ be regular cardinals. An ideal I on $\mathcal{P}_\kappa\lambda$ is precipitous iff player I does not have a winning strategy in $G(I)$ (iff player II has a winning strategy in $G(I)$).*

(For a proof of this equivalence, see, for example, [M], p. 111. The equivalence shown there uses a version of the game where player I does not play functions; that version, however, can easily be seen to be equivalent to the one used here.)

Denote the NS ideal on κ by NS_κ. We will now show how to make NS_κ non-precipitous. The following theorem is due to Foreman, Magidor and Shelah. See

[FMS], Theorem 33, p. 41. The proof I will present, which is a simplification of the original proof, is due to Velickovic. It turns out that if we force with κ^+ copies of the forcing notion P^1_κ (the notion of forcing used to destroy presaturated ideals on κ), the NS ideal on κ is not precipitous in the generic extension.

THEOREM 3.2. *Let $V \models \kappa^{<\kappa} = \kappa$. Let $P^3_\kappa = \prod_{\alpha\in\kappa^+} P_\alpha$, where each P_α is isomorphic to P^1_κ (as defined in section 2), and the product is taken with $< \kappa$ supports. Let G be a generic filter over P^3_κ. Then in $V[G]$, NS_κ is not precipitous, and no cardinals are collapsed.*

PROOF. We begin by showing no cardinals are collapsed. Again, P^3_κ is $< \kappa$-closed, so no cardinal $\le \kappa$ is collapsed. We will now argue that P^3_κ has the κ^+-c.c., and thus the proof that no cardinals are collapsed will be complete.

Assume A is an antichain in $P^3_\kappa, |A| = \kappa^+$. Since support$(p)$ has size less than κ for each $p \in P$, we must have, using the Δ system lemma (see, e.g., [Kun], p. 49, theorem 1.6), a set $B \subset A$, $|B| = \kappa^+$ such that $\forall p,q \in B$, support$(p) \cap$ support$(q) = r$. Now $p,q \in P^3_\kappa$ can be incompatible only if there is an $\alpha \in$ support$(p) \cap$ support(q) and $p(\alpha)$ is incompatible with $q(\alpha)$ in P_α. So we only have to worry about $\alpha \in r$.

For $p \in P^3_\kappa$, let $p(\alpha) = \langle p(\alpha)_1, p(\alpha)_2\rangle$. Since $|r| < \kappa$, and for each $\alpha \in r$ and $p \in B$ there are only κ many possibilities for $p(\alpha)_1$, there are only $\kappa^{<\kappa} = \kappa$ possibilities for $\langle p(\alpha)_1 \,|\, p \in B, \alpha \in r\rangle$. Thus there must be a set $C \subset B$, $|C| = \kappa^+$ such that $\forall p,q \in C$, $\forall \alpha \in r\ p(\alpha)_1 = q(\alpha)_1$. But then (as before, in proof of Claim 2.2), any two members of C must be compatible. This is a contradiction. Thus P^3_κ has the κ^+-c.c.

Before we continue with the proof we would like to note that since P^3_κ is $< \kappa$-closed and each P_α has conditions of size $< \kappa$ which, essentially, have ordinals as members, we can think of P^3_κ either as a product (which is how we define P^3_κ) or as an iteration. We will use this duality below.

Let G be a generic filter over P^3_κ. We will now prove that in $V[G]$, NS_κ is not precipitous. We will do this by describing a winning strategy for player I in $G(NS_\kappa)$.

Assume each player has made n moves in $G(NS_\kappa)$. We will describe player I's $n+1$st move. In general, player I will use the generic functions one gets when forcing with P^3_κ as his f's. Let g_α be the generic function corresponding to P_α. Let $\theta < \kappa^+$ be the least such that $A_1, \ldots, A_n, B_1, \ldots, B_n, f_1, \ldots, f_n \in V[G_\theta]$ (where $G_\theta = G \restriction \prod_{\alpha\in\theta} P_\alpha$). Player I should play $g_{\theta+1}$ as f_{n+1} and $\{\alpha \in B_n \,|\, g_{\theta+1}(\alpha) < f_n(\alpha)\}$ as his A_{n+1}.

We have now described player I's $n+1$st move. We have to prove it is a "legal" move in the game. Clearly $A_{n+1} \subset B_n$ and we made sure $\forall \alpha \in A_{n+1} f_{n+1}(\alpha) < f_n(\alpha)$; what we have to prove is that A_{n+1} is stationary.

Assume not. We work in $V[G_\theta]$, thus the names we pick will be $P^3_\kappa/\prod_{\alpha\in\theta} P_\alpha$ names, and the conditions we pick will belong to $P^3_\kappa/\prod_{\alpha\in\theta} P_\alpha$. Also, the forcing will take place over $V[G_\theta]$. Let τ be a name for a club subset of κ (of $V[G]$) and let $p \in G$ be such that

$$p \Vdash \tau \text{ is a club} \quad \text{and} \quad p \Vdash \underline{A_{n+1}} \cap \tau = \emptyset$$

where $\underline{A_{n+1}}$ is a name for A_{n+1}.

Let λ be a regular cardinal, $\lambda > 2^{2^\kappa}$, Δ a well ordering of $H_\lambda^{V[G_\theta]}$. Let $M \prec \langle H_\lambda^{V[G_\theta]}, \in, \Delta, \kappa, P^3_\kappa, \theta, G_\theta \rangle$ be such that:

(a) $|M| = \kappa, M \cap \kappa^+ \in \mathrm{ORD}$ and $M^{<\kappa} \subset M$.

(b) $p, \tau \in M$.

Let $\delta^* = M \cap \kappa^+$. Let $h : \kappa \to \delta^*$ enumerate δ^*. (We can take $h \in V$.)

Let $\langle N_\alpha \mid \alpha < \kappa \rangle$ be a continuous chain of elementary substructures of M, each of cardinality $< \kappa$ such that:

(a) $M = \bigcup_{\alpha<\kappa} N_\alpha$.

(b) $p, \tau \in N_0$.

(c) $\langle N_\beta \mid \beta < \alpha \rangle \in N_{\alpha+1}$.

Note that there is a club $C \subset \kappa$ such that $\alpha \in C \Rightarrow N_\alpha \cap \kappa = \alpha$. Since each α in C is a limit ordinal, clause (c) guarantees that $\alpha \in C \Rightarrow N_\alpha$ is internally approachable. (See [FMS] for a definition of internally approachable.)

To get a contradiction (to the assumption that $p \Vdash \underline{A_{n+1}} \cap \tau = \emptyset$) we want to find a $q \leq p$ and an $\alpha \in \kappa$ such that $q \Vdash \check{\alpha} \in \underline{A_{n+1}} \cap \tau$.

We will first pick α and only then construct a suitable q for it. However, α will be picked so that it will be easy to construct a suitable q for it.

Since $A_{n+1} = \{\alpha \in B_n \mid f_n(\alpha) > g_{\theta+1}(\alpha)\}$, we will pick $\alpha \in B_n$. We also want to make sure $f_n(\alpha)$ is big enough, so that we will be able to guarantee $f_n(\alpha) > g_{\theta+1}(\alpha)$. We do this by making sure that $f_n(\alpha) > O_{\delta^*}(\alpha)$. Let

$$C_1 = \{\alpha < \kappa \mid f_n(\alpha) > O_{\delta^*}(\alpha)\}.$$

Note that C_1 is club in κ, since f_n is one of the generic functions added by our forcing and these functions dominate (eventually) *all* O_β functions. In order to make use of the enumerating function h, we need to guarantee that N_α, for the α we pick, is closed under h, h^{-1}. Thus let

$$C_2 = \{\alpha < \kappa \mid N_\alpha \text{ is closed under } h, h^{-1}\},$$

C_2 is club in κ. Finally, we require also that α belong to C_3, where

$$C_3 = \{\alpha \in C \mid \forall\beta \in \alpha\ O_{\delta^*}(\alpha) > O_{h(\beta)}(\alpha)\}.$$

This will help us find a suitable q; more specifically, we will use it to show that q is stronger than some given conditions, as will become clearer below. C_3 contains a club in κ.

So let $\alpha \in C_1 \cap C_2 \cap C_3 \cap B_n$. We now want to find a $q \le p$ such that $q \Vdash \check{\alpha} \in \underline{A_{n+1}} \cap \tau$.

We picked α so that $\alpha \in B_n$; so for a condition q to force α in A_{n+1}, all we have to guarantee is that $q \Vdash (f_n(\alpha))^{\vee} > \underline{g_{\theta+1}}(\check{\alpha})$ ($\underline{g_{\theta+1}}$ is a name for $g_{\theta+1}$). Recall that we are working in $V[G_\theta]$, where $g_{\theta+1}$ is not yet defined. Thus we can make q force $f_n(\alpha) > g_{\theta+1}(\alpha)$ if we guarantee that q determines the value of $g_{\theta+1}(\alpha)$ to be smaller than $f_n(\alpha)$. Such a value is $O_{\delta^*}(\alpha)$, since α is in C_1. Thus if we can arrange that $\langle\alpha, O_{\delta^*}(\alpha)\rangle$ belongs to the first half of the $\theta + 1$st coordinate of q (i.e., $\langle\alpha, O_{\delta^*}(\alpha)\rangle \in q(\theta + 1)_1$) then we will have that $q \Vdash \check{\alpha} \in \underline{A_{n+1}}$.

Assume some condition, q', forces α to be in τ, i.e., $q' \Vdash \check{\alpha} \in \tau$. We can then *define* q as follows:

$$q(\alpha) = q'(\alpha) \qquad \text{for all } \alpha \neq \theta + 1,$$

$$(*) \qquad q(\theta + 1)_1 = q'(\theta + 1)_1 \cup \{\langle\alpha, O_{\delta^*}(\alpha)\rangle\},$$

$$q(\theta + 1)_2 = q'(\theta + 1)_2,$$

provided q' does not yet determine a value for $g_{\theta+1}(\alpha)$, i.e., provided there is no β such that $\langle\alpha,\beta\rangle$ belongs to $q'(\theta + 1)_1$. It will be the case that q is stronger than q' (and thus forces α to be in τ) iff $O_{\delta^*}(\alpha) > O_\beta(\alpha)$ for all β mentioned in the second part of the $\theta + 1$st coordinate of q'.

Now assume we can find a condition q' such that $q'(\theta + 1) \subset N_\alpha$ and such that $q' \Vdash \check{\alpha} \in \tau$. Then both the above-mentioned difficulties can be overcome. First, if $q'(\theta + 1) \subset N_\alpha$, then $\langle\alpha,\beta\rangle \notin q'(\theta + 1)_1$, for any β, since otherwise we would get that $\alpha \in N_\alpha$, contradiction (recall that $N_\alpha \cap \kappa = \alpha$). Second, if $\beta \in q'(\theta + 1)_2$ then $\beta \in N_\alpha$. So if $O_{\delta^*}(\alpha)$ is greater than $O_\beta(\alpha)$ for all $\beta \in N_\alpha$, then q will be stronger than q'. But if $\kappa \le \beta < \kappa^+$, $\beta \in N_\alpha$ then $\beta = h(\sigma)$, for some $\sigma \in \alpha$, since N_α is closed under h, h^{-1}. Now recall that $\alpha \in C_3$, so that $O_{\delta^*}(\alpha) > O_{h(\gamma)}(\alpha)$ for all $\gamma \in \alpha$, and so $O_{\delta^*}(\alpha) > O_{h(\sigma)}(\alpha)$, and hence $O_{\delta^*}(\alpha) > O_\beta(\alpha)$, which is exactly what we want.

Thus if q' is a condition such that $q'(\theta + 1) \subset N_\alpha$ and $q' \Vdash \check{\alpha} \in \tau$, we can define q using (*) and we will have that $q \Vdash \check{\alpha} \in \tau \cap \underline{A_{n+1}}$, which is our goal. Thus it only remains to show that we can construct a q' with $q'(\theta + 1) \subset N_\alpha$ and $q' \Vdash \check{\alpha} \in \tau$.

One can argue from general facts (e.g., that N_α is internally approachable) that it is possible to construct a master condition for N_α (which is a union of conditions in N_α); such a condition would meet our needs (for a definition of a "master condition" and a proof of this fact, see [FMS]). However, since the construction of a suitable condition is fairly simple, we describe it here.

Define a sequence of conditions $\langle q_\beta \mid \beta < \alpha \rangle$ and a sequence of ordinals $\langle \sigma_\beta \mid \beta < \alpha \rangle$ with the following properties:

(a) $\beta > \gamma \Rightarrow q_\beta \leq q_\gamma$,

(b) $q_\beta \in N_{\beta+2}$,

(c) $\sigma_\beta \geq N_\beta \cap \kappa$,

(d) $q_\beta \Vdash \check{\sigma}_\beta \in \tau$,

(e) q_β is definable in $N_{\beta+2}$ from N_β and $\langle q_\gamma \mid \gamma < \beta \rangle$.

We first show this construction is possible, and then argue that if we let $q' = \bigcup_{\beta \in \alpha} q_\beta$ then q' is a condition which meets our needs. To show that the construction is possible, assume we have $\langle q_\gamma \mid \gamma < \beta \rangle$, $\langle \sigma_\gamma \mid \gamma < \beta \rangle$, with the above properties. By (e), we have that $\langle q_\gamma \mid \gamma < \beta \rangle \in N_{\beta+2}$, since $\langle N_\gamma \mid \gamma < \beta \rangle \in N_{\beta+2}$. Thus in $N_{\beta+2}$, we can let q_β be the least (with respect to Δ) condition which is stronger than each condition in $\langle q_\gamma \mid \gamma < \beta \rangle$ and which forces some ordinal greater than $N_\beta \cap \kappa$ to be in τ (such a condition exists since $p \Vdash \tau$ is club and $p, \tau \in N_0$). Let σ_β be the least such ordinal. Then q_β, σ_β satisfy the required condition at stage β. Thus such a construction can be carried through.

We can now define $q' = \bigcup_{\beta \in \alpha} q_\beta$. Note first that since $\theta + 1 \in N_\alpha$, $q_\beta(\theta + 1) \in N_\alpha$ for all $\beta \in \alpha$. Also, since $|q_\beta(\theta + 1)| < \kappa$, we have that $N_\alpha \vDash |q_\beta(\theta + 1)| < \kappa$. Thus there is a function f in N_α such that f maps some ordinal in $N_\alpha \cap \kappa = \alpha$, say γ_β, onto $q_\beta(\theta + 1)$. Since $\gamma_\beta \subset N_\alpha$, $f(\delta) \in N_\alpha$ for all $\delta < \gamma_\beta$. So $q_\beta(\theta + 1) \subset N_\alpha$ for all $\beta < \alpha$ and thus $q'(\theta + 1) \subset N_\alpha$. Second, it is clear from the definition of q' that $q' \Vdash \check{\alpha} \in \tau$. Thus q' satisfies all the requirements that we were after. As explained above, we can now define q from q' using (*) and get a contradiction to the assumption that $p \Vdash \underline{A_{n+1}} \cap \tau = \emptyset$. This completes the proof. ■

Remark. In fact, we can see that in the model $V[G]$ described above, more ideals besides NS_κ were made non-precipitous. These fall into two categories:

(1) Restrictions of NS_κ: Given an ideal I on $\mathcal{P}_\kappa\lambda$ and an I positive set X, we define the restriction of I to X ($I \restriction X$) to be $\{Y \in \mathcal{P}_\kappa\lambda \mid Y \cap X \in I\}$. Forcing with $(I \restriction X)^+$ is the same as forcing with I^+ below X.

In the model $V[G]$ described above, we have that $\forall X \subset \kappa$, X stationary, $NS_\kappa \restriction X$ is not precipitous. This fact is immediate from the proof of the previous theorem, since player I had *no* restrictions on his first move. If we wanted to prove $NS_\kappa \restriction X$ is not precipitous, we would have player I play X as his A_0.

(2) Here we consider a different generalization of NS_κ. NS_κ in $V[G]$ is the *normal closure* of NS_κ in V: i.e., it is the smallest normal ideal containing $(NS_\kappa)^V$. If P is any forcing notion that is κ-closed, and G a generic filter over P, then the normal closure of a proper ideal in V is always a proper ideal in $V[G]$. See [FMS], lemma 34, p. 42. It turns out that in the model constructed above, not only is the NS ideal on κ_1 made non-precipitous, but that, in fact, all normal closures of ideals on κ_1 in V are made not precipitous in $V[G]$. This requires a slight modification of the proof. See [FMS].

We now generalize the previous result to $\mathcal{P}_\kappa\lambda$, i.e., for any regular κ and λ ($\kappa \leq \lambda$) show how to get a model where the NS ideal on $\mathcal{P}_\kappa\lambda$ is not precipitous.

For the construction of the model we will need analogues of the O_β functions we had before. We will call them L_β functions. For $\mathcal{P}_\kappa\lambda$ we will have $\langle L^\kappa_\beta \mid \lambda \leq \beta < \lambda^+\rangle$, $L^\kappa_\beta : \mathcal{P}_\kappa\lambda \to \lambda$. The main property we would like these L^κ_β's to have is: Given any set $B \subset \lambda^+ \backslash \lambda$, $|B| < \lambda$ and any $\alpha > \sup B$, $\lambda \leq \alpha < \lambda^+$, the set

$$\{X \in \mathcal{P}_\kappa\lambda \mid \forall \beta \in B\ L^\kappa_\alpha(X) > L^\kappa_\beta(X)\}$$

contains a club in $\mathcal{P}_\kappa\lambda$.

This is the reason we cannot use the "natural" analogues of the O_β functions used before, namely, the O'_β functions mentioned in the proof of Theorem 2.3: The use of these functions would restrict us to $|B| < \kappa$. Instead, we will use the $< \lambda$-closure of club sets in λ: Let $\langle O_\beta \mid \lambda \leq \beta < \lambda^+\rangle$ be the O_β functions defined before (for λ). Define $\langle L^\kappa_\beta \mid \lambda \leq \beta < \lambda^+\rangle$, $L^\kappa_\beta : \mathcal{P}_\kappa\lambda \to \lambda$ as follows:

$$L^\kappa_\beta(X) = O_\beta(\sup X) \qquad (X \in \mathcal{P}_\kappa\lambda).$$

We will first show that the L^κ_β functions have the property we wanted them to have: Let $B \subset \lambda^+ \backslash \lambda$, $|B| < \lambda$, $\alpha > \sup B$, $\alpha < \lambda^+$. Let $A = \{X \mid \forall \beta \in B\ L^\kappa_\alpha(X) > L^\kappa_\beta(X)\}$. We want to show that A contains a club. Let $C = \{\delta \in \lambda \mid \forall \beta \in B\ O_\alpha(\delta) > O_\beta(\delta)\}$. Then C contains a club C' in λ. We will show that $A' = \{X \in A \mid \sup X \in C'\}$ is club. First, assume $X \in \mathcal{P}_\kappa\lambda$. We want to find a $Y \in A'$, $X \subset Y$. Pick γ such that $\gamma \in C'$, $\operatorname{cof}\gamma < \kappa$ and $\gamma > \sup X$. Then pick $Y \supset X$, $\sup Y = \gamma$ ($Y \in \mathcal{P}_\kappa\lambda$). Since $\gamma \in C'$, we have that $\forall \beta \in B\ O_\alpha(\gamma) > O_\beta(\gamma)$. Thus, $\forall \beta \in B\ O_\alpha(\sup Y) > O_\beta(\sup Y)$. Thus, $\forall \beta \in B\ L^\kappa_\alpha(Y) > L^\kappa_\beta(Y)$. Thus $Y \in A'$ and hence A' is unbounded in $\mathcal{P}_\kappa\lambda$.

It is also easy to see that A is closed. If $\langle X_i \mid i \in \gamma\rangle$ is an increasing sequence of sets in A' of length less than κ (i.e., $\gamma < \kappa$) then for all i in γ, $\sup X_i \in C$. But then $\bigcup_{i\in\gamma} \sup X_i = \sup(\bigcup_{i\in\gamma} X_i) \in C'$ so that $\bigcup_{i\in\gamma} X_i \in A'$.

We can now describe the forcing notion we will use. In order to make the NS ideal on $\mathcal{P}_\kappa\lambda$ not precipitous, we will force with λ^+ copies of the forcing notion that adds a function from $\mathcal{P}_\kappa\lambda$ to λ which dominates eventually (in the same sense as in the proof of Theorem 2.3) all $\langle L^\kappa_\beta \mid \lambda \leq \beta < \lambda^+\rangle$ functions. The forcing will be with $< \lambda$-supports.

The forcing P for adding *one* function that will eventually dominate all L^κ_β functions is with conditions of size less than λ, where each $p \in P$ has two parts $p = \langle p_1, p_2\rangle : p_1$ is a partial function from $\mathcal{P}_\kappa\lambda$ into λ, and p_2 is a set of ordinals $< \lambda^+$. As for the order on $P: q \leq p$ iff $p_1 \subset q_1$ and $\langle X,\beta\rangle \in q_1 \setminus p_1 \Rightarrow \beta > L^\kappa_\gamma(X)$ for all $\gamma \in p_2$.

THEOREM 3.3. *Let $\kappa \leq \lambda$ be uncountable regular cardinals satisfying $\lambda^{<\lambda} = \lambda$. Let $P^4_{\kappa,\lambda} = \prod_{\alpha<\lambda^+} P^\kappa_\alpha$, where each P^κ_α is (isomorphic to) the notion of forcing described above for adding a function dominating all $\langle L^\kappa_\beta \mid \lambda \leq \beta < \lambda^+\rangle$. The product is taken with $< \lambda$-supports. Let G be a generic filter over $P^4_{\kappa,\lambda}$. Then in $V[G]$, the NS ideal on $\mathcal{P}_\kappa\lambda$ is not precipitous (and no cardinals are collapsed).*

PROOF. The proof that for each α, P^κ_α has the λ^+-c.c. and that the product, $\prod_{\alpha<\lambda^+} P^\kappa_\alpha$ (with $< \lambda$-supports), has the λ^+-c.c. is exactly as before (as in proving the analogous facts for P^3_λ). Also, each P^κ_α, and thus $P^4_{\kappa,\lambda}$, is $< \lambda$-closed. Thus cardinals are not collapsed when forcing with $P^4_{\kappa,\lambda}$.

The proof that the NS ideal on $\mathcal{P}_\kappa\lambda$ in $V[G]$ is not precipitous follows the same general lines as before, but certain changes have to be made. We will point out where these alterations take place. Before continuing with the proof, we mention the following relationship between stationary (and club) subsets of $\mathcal{P}_\kappa X$ and $\mathcal{P}_\kappa\lambda$, when X is a set containing λ, which we will have to rely on:

If S is a stationary (club) subset of $\mathcal{P}_\kappa X$ then

$$S^* = \{Y \cap \lambda \mid Y \in S\}$$

is stationary (contains a club) in $\mathcal{P}_\kappa\lambda$.

Conversely, if S is a stationary (club) subset of $\mathcal{P}_\kappa\lambda$ then

$$\bar{S} = \{Y \in \mathcal{P}_\kappa X \mid Y \cap \lambda \in S\}$$

is a stationary (club) subset of $\mathcal{P}_\kappa X$.

A proof of these facts can be found in [FMS].

We now go back to the proof of Theorem 3.3. We want to describe a strategy for player one in G (NS ideal on $\mathcal{P}_\kappa\lambda$). Player I again plays generic functions for his function moves, and we want to show that

$$A_{n+1} = \{X \in B_n \mid g_{\theta+1}(X) < f_n(X)\}$$

is stationary, where θ is the least such that $A_1, \ldots, A_n, B_1, \ldots, B_n, f_1, \ldots, f_n \in V[G_\theta]$.

Assume it is not. We now work in $V[G_\theta]$ (as in the proof of Theorem 3.2, we will now pick $P^4_{\kappa,\lambda}/\prod_{\alpha\in\theta} P^\kappa_\alpha$ names, etc.) and let $p \Vdash \underline{A_{n+1}} \cap \tau = \emptyset$ and $p \Vdash \tau$ is club in $(\mathcal{P}_\kappa\lambda)^V$. As before, we will reach a contradiction by finding an X in $\mathcal{P}_\kappa\lambda$ and a q stronger than p such that $q \Vdash X \in \underline{A_{n+1}} \cap \tau$. However, in this case, we need to work in $V[G]$ at this point.

We now work in $V[G]$. Let $M < \langle H^{V[G]}_\sigma, \in, \Delta, \kappa, \lambda, P^4_{\kappa,\lambda}, G, \theta\rangle$, where $\sigma > 2^{2^\lambda}$, Δ is a well ordering of $H^{V[G_\theta]}_\sigma$, and such that:

(a) $|M| = \lambda, M \cap \lambda^+ \in \mathrm{ORD}$, $M^{<\lambda} \subset M$.

(b) $p, \tau \in M$.

Let $\delta^* = M \cap \lambda^+$. Let h enumerate all $< \lambda$ sequences of ordinals $< \lambda^+$ in M. So $h : \lambda \to (M \cap \lambda^+)^{<\lambda}$. (There is such an h since $|M| = \lambda$ and $\lambda^{<\lambda} = \lambda$. We pick h in $V[G]$.)

Let

$$C_1 = \{N < M \mid |N| < \kappa; p, \tau, \lambda \in N, N \text{ is closed under } h, h^{-1} \text{ and } N \cap \kappa \in \kappa\}.$$

C_1 is club in $P_\kappa M$. Let

$$C_1^* = \{N \cap \lambda \mid N \in C_1\}.$$

C_1^* contains a club in $P_\kappa\lambda$. Let

$$C_2 = \{X \mid f_n(X) > L^\kappa_{\delta^*}(X)\}.$$

C_2 contains a club in $\mathcal{P}_\kappa\lambda$. Let

$$C_3 = \{X \in C_1^* \mid \forall\beta \in X\ \forall\alpha \in h(\beta) L^\kappa_{\delta^*}(X) > L^\kappa_\alpha(X)\}.$$

C_3 contains a club in $\mathcal{P}_\kappa\lambda$. To see that, let

$$D_\beta = \{X \in C_1^* \mid \forall\alpha \in h(\beta)\ L^\kappa_{\delta^*}(X) > L^\kappa_\alpha(X)\} \qquad (\beta \in X).$$

Then D_β contains a club because $|h(\beta)| < \lambda$; but $C_3 = \Delta_{\beta<\lambda} D_\beta$, so C_3 is club.

Let $X \in C_2 \cap C_3 \cap B_n$. Note that this is possible, since B_n is a set which is known to be stationary in $V[G]$ (it is a legal move in the game, which takes place in $V[G]$.) Then $X = N \cap \lambda$ for some $N \in C_1$. Let P^* be $P/\prod_{\alpha\in\theta} P^\kappa_\alpha$ and G^* be

G/G_θ. Since N belongs to C_1, we have that N is an elementary substructure of $\langle H_\sigma^{V[G]}, \in, \Delta, P^4_{\kappa,\lambda}, G, \theta\rangle$ and thus $N \models$ "G^* meets every dense set of P^* which belongs to $V[G_\theta]$", i.e., G^* meets $D \cap N$ for all dense sets D (in P^*) in $V[G_\theta]$ which belong to N.

Let T be $\{t \in N \mid t \in G^*\}$ and let q' be the union of T. We know that q' is a condition (in P^*) since $|T| < \kappa \leq \lambda$, and all $t \in T$ are compatible, as they all belong to G^*. We now want to argue that $q' \Vdash N \cap \lambda \in \tau$. Let $A = \{X \in \mathcal{P}_\kappa\lambda \cap N \mid q' \Vdash X \in \tau\}$. We claim A is a directed system of sets in $\mathcal{P}_\kappa\lambda$ of size less than κ, whose union is $N \cap \lambda$. First, A has size less than κ, since it is contained in N. Second, $\bigcup A \subset N \cap \lambda$, since every member of $\mathcal{P}_\kappa\lambda$ in N is also a subset of N (recall that $N \cap \kappa$ is an ordinal). Also, for all $\alpha \in N \cap \lambda$ there is an X in $\mathcal{P}_\kappa\lambda$ such that $\alpha \in X \in \tau_G$. By elementarity of N, there is such an X in N. If $X \in \tau_G$, there is a condition in G which forces $X \in \tau$. If $X \in N$, there is such a condition in N (and hence in $N \cap G$). Thus for every α in $N \cap \lambda$ there is an X in $N \cap \mathcal{P}_\kappa\lambda$ such that $\alpha \in X$ and $q' \Vdash X \in \tau$. Hence the union of A is $N \cap \lambda$. The argument that A is directed is similar and we omit it here. Thus $q' \Vdash N \cap \lambda \in \tau$.

N itself is a member of $V[G]$; however, $N \cap V[G_\theta]$ is a member of $V[G_\theta]$, since $|N| < \kappa \leq \lambda$ and the forcing (P^*) is $< \lambda$ closed. Let $K = N \cap V[G_\theta]$. Then q' is a union of conditions in K $(T \subset P^*)$ and $q' \Vdash_{P^*} \check{X} \in \tau$ (recall that $X = N \cap \lambda = K \cap \lambda$). Thus, even though we found N in $V[G]$, we are now back to the situation we had in the "one-dimensional" case, namely, in $V[G_\theta]$, we have a condition, q', and a set, K, such that $q' \Vdash (K \cap \lambda)^\vee \in \tau$ and q' is a union of conditions in K (although this union belongs to $V[G]$, not necessarily $V[G_\theta]$).

We now continue the proof along the lines of the proof for the "one-dimensional" (ideals on κ) case, but we move back to $V[G]$. Define q to be exactly the same as q', except that at the $\theta + 1$st coordinate we add $\langle X, L^\kappa_{\delta^*}(X)\rangle$. A little more care than before has to go into showing that

(a) q is a condition,

(b) q is stronger than q'.

The reason is that, before, the analogous claims followed from the fact that $q'(\theta + 1)$ was a *subset* of N_α. Here we cannot hope to show that $q'(\theta + 1)$ is a subset of N, because N has cardinality less than κ and $q'(\theta + 1)$ may have larger cardinality. However, we can still show that (a) and (b) hold. First, to argue that q is a condition, we need to show that $\langle X, \gamma\rangle \notin \mathrm{dom}(q'(\theta + 1)_1)$, for any γ. We will show that $\langle X, \gamma\rangle \notin \mathrm{dom}(r(\theta + 1)_1)$, for any r in N. Since q' is a union of conditions belonging to N, this will show q' has the desired property. So let $r \in N$. Then $a = \bigcup(\mathrm{dom}(r(\theta + 1)_1)) \in N$. If $X = N \cap \lambda \in \mathrm{dom}(r(\theta + 1)_1)$, then $N \cap \lambda \subset a$. Also, $a \neq \lambda$, since $|r(\theta + 1)_1| < \lambda$. But then $N \models \forall\beta < \lambda\ \beta \in a$ so that

$M \models \forall\beta < \lambda\ \beta \in a$ so that $\lambda \subset a$, contradiction. Thus $X \notin \mathrm{dom}(q'(\theta + 1)_1)$ and we are free to add $\langle X, L^{\kappa}_{\delta^*}(X)\rangle$ to it, i.e., q is a condition (in P^*).

Second, to show that $q \le q'$, we need to show that $L^{\kappa}_{\delta^*}(X) > L^{\kappa}_{\alpha}(X)$ for all α mentioned in $q(\theta + 1)_2$. Not every such ordinal is a member of N, but each such ordinal is a member of a *sequence* of ordinals, namely $t(\theta + 1)_2$ (for some t in N), which *is* a member of N. For every such t, $t(\theta + 1)_2$ is a $< \lambda$ sequence of ordinals $< \lambda^+$ and thus in the range of h (recall that h enumerates all $< \lambda$ sequences of ordinals $< \lambda^+$ in M). Since N is closed under h^{-1}, there is an ordinal β in N such that $t(\theta + 1)_2 = h(\beta)$. Thus if $\alpha \in q(\theta + 1)_2$, there is a β in N such that $\alpha \in h(\beta)$. Since $X \in C_3$, this guarantees that $L^{\kappa}_{\delta^*}(X) > L^{\kappa}_{\alpha}(X)$. Thus $q \le q'$.

We can now conclude the argument as before: Since $q \le q'$, $q \Vdash \check{X} \in \tau$. Our construction of q guaranteed that $q \Vdash g_{\theta+1}(\check{X}) = \check{L}^{\kappa}_{\delta^*}(\check{X})$, and since $X \in C_2 \cap B_n$, it follows that $q \Vdash \check{X} \in \underline{A_{n+1}}$. Hence $q \Vdash \check{X} \in \tau \cap \underline{A_{n+1}}$, contradiction. ■

REMARK. The remark at the end of Theorem 3.2.4 applies here, as well, namely, in the model described above, making the NS ideal on $\mathcal{P}_\kappa\lambda$ not precipitous, all restrictions of the NS ideal on $\mathcal{P}_\kappa\lambda$ are likewise not precipitous, as well as all normal closures of ideals in V in $\mathcal{P}_\kappa\lambda$.

4. In this section we show that the projection of a precipitous ideal on $\mathcal{P}_{\omega_1}\lambda$ (onto $\mathcal{P}_{\omega_1}\lambda'$, $\lambda' < \lambda$ uncountable regular cardinals) need not itself be precipitous. For the results in this section and the next, we need the following theorem:

THEOREM 4.1. *Let δ be a Woodin cardinal. Let $\lambda < \delta$ be a regular uncountable cardinal. Let G be generic over* $\mathrm{Lv}(<\delta, \lambda)$, *the forcing notion for collapsing cardinals less than δ to λ with conditions of size less than λ. Then in $V[G]$, the NS ideal on $\mathcal{P}_{\omega_1}\lambda$ is presaturated (and thus is also precipitous).*

This theorem, with δ being *supercompact*, is due to Foreman–Magidor–Shelah. For a proof that this ideal is precipitous see [FMS], Theorem 29. The reduction of δ being Woodin is due to the author. For a full proof of the present theorem, see [G]. For a discussion of Woodin cardinals, see [MS].

We now go back to constructing a model where the projection of a precipitous ideal is not precipitous. Our ideal I on $\mathcal{P}_{\omega_1}\lambda$ will be the NS ideal. In this case, the projection of I onto $\mathcal{P}_{\omega_1}\lambda'$ is simply the NS ideal on $\mathcal{P}_{\omega_1}\lambda'$ (this follows from the discussion in the proof of Theorem 3.3 of the relationship between stationary subsets of $\mathcal{P}_{\omega_1}\lambda$ and those of $\mathcal{P}_{\omega_1}\lambda'$). Thus it is enough to construct a model where the NS ideal on $\mathcal{P}_{\omega_1}\lambda$ is precipitous but the one on $\mathcal{P}_{\omega_1}\lambda'$ is not.

THEOREM 4.2. *Let δ be a Woodin cardinal, $\lambda' < \lambda < \delta$ uncountable regular cardinals. Assume $\lambda'^{<\lambda'} < \lambda$. Then there is a forcing notion P, $|P| = \delta$, such that if G is a generic filter over P, then in $V[G]$ the NS ideal on $\mathcal{P}_{\omega_1}\lambda$ is precipitous, but the NS ideal on $\mathcal{P}_{\omega_1}\lambda'$ is not.*

PROOF. We may assume that $\lambda'^{<\lambda'} = \lambda'$, since this can always be made true by small forcing which does not affect the Woodiness of δ, the regularity of λ', λ or the fact that $\lambda' < \lambda$ (since $\lambda'^{<\lambda'} < \lambda$).

Let $P^4_{\omega_1,\lambda'}$ be the forcing described in the previous section which turns the NS ideal on $\mathcal{P}_{\omega_1}\lambda'$ into a non-precipitous ideal (see Theorem 3.3). Let G_1 be generic over $P^4_{\omega_1,\lambda'}$. Then

$$V[G_1] \models \text{the NS ideal on } P_{\omega_1}\lambda' \text{ is not precipitous.}$$

But δ is still Woodin in $V[G_1]$, since $|P^4_{\omega_1,\lambda'}| = ((\lambda')^+)^{\lambda'}$; so if we let G_2 be $V[G_1]$ generic over $\mathrm{Lv}(<\delta,\lambda)$, then

$$V[G_1][G_2] \models \text{the NS ideal on } \mathcal{P}_{\omega_1}\lambda \text{ is precipitous,}$$

by Theorem 4.1. Thus we will be done once we show that the NS ideal on $\mathcal{P}_{\omega_1}\lambda'$ remains non-precipitous. We need the following lemma:

LEMMA 4.3. *Let $M \subset V$ be models of ZFC, let $\kappa < \lambda$ be regular cardinals in M and in V. Suppose $(\mathcal{P}_\kappa\lambda)^M = (\mathcal{P}_\kappa\lambda)^V$ and $\mathcal{P}^M(\mathcal{P}_\kappa\lambda) = \mathcal{P}^V(\mathcal{P}_\kappa\lambda)$. Let I be an ideal on $\mathcal{P}_\kappa\lambda$ in M. Let $M \models I$ be not precipitous. Then $V \models I$ is not precipitous.*

PROOF. $(I^+)^M = (I^+)^V$. (I^+ are the positive sets with respect to I.) Since $M \models I$ is not precipitous, $\exists S \in I^+$ such that

$$S \Vdash M^{P_\kappa\lambda}/\Gamma \text{ is ill founded,}$$

where Γ is a name for the generic object over I^+. Let G be a V-generic filter over I^+ such that $S \in G$. Then G is also M-generic over I^+.

Since $S \in G$, $M^{P_\kappa\lambda}/G$ is ill founded; say $\langle f_i \mid i \in \omega\rangle$, $f_i : \mathcal{P}_\kappa\lambda \to \mathrm{ORD}$, is a sequence of functions in $M[G]$ which generates a descending sequence in the ultrapower, i.e., $[f_1] >_G [f_2] >_G \cdots$. But $f_i \in M$ for all $i \in \omega$, so $f_i \in V$ for all $i \in \omega$. The sequence $\langle f_i \mid i \in \omega\rangle$ is in $M[G]$ and thus in $V[G]$. Hence $\langle f_i \mid i \in \omega\rangle$ witnesses the ill-foundedness of $V^{P_\kappa\lambda}/G$. Thus I is not precipitous in V. ■

Now $\mathrm{Lv}(<\delta,\lambda)$ is $<\lambda$-closed, and thus, since $\lambda' < \lambda$ and $\lambda'^{<\lambda'} = \lambda'$, no new subsets of $\mathcal{P}_{\omega_1}\lambda'$ were added. Thus we can apply the above lemma and conclude that the NS ideal on $\mathcal{P}_{\omega_1}\lambda'$ remains non-precipitous in $V[G_1][G_2]$. Thus $V[G_1][G_2]$ is the model we were looking for. ■

NOTE. By Theorem 4.1, the NS ideal on $\mathcal{P}_{\omega_1}\lambda$ in $V[G_1][G_2]$ is, in fact, presaturated. Thus we have proved:

THEOREM 4.4. *Let δ be a Woodin cardinal (in V), and $\lambda' < \lambda < \delta$ uncountable regular cardinals satisfying $\lambda'^{<\lambda'} < \lambda$. Then there is a forcing notion P, $|P| = \delta$, such that if G is generic over P then in $V[G]$ the NS ideal on $\mathcal{P}_{\omega_1}\lambda$ is presaturated, but the NS ideal on $\mathcal{P}_{\omega_1}\lambda'$ is not precipitous.*

In this paper, we have concentrated on ideals on $\mathcal{P}_{\omega_1}\lambda$. However, one can generalize the results in this section to ideals on $\mathcal{P}_\kappa\lambda$, for κ an arbitrary uncountable regular cardinal (less than or equal to λ). First, we have:

THEOREM 4.5. *Let δ be a supercompact cardinal (in V), and $\kappa \leq \lambda' < \lambda < \delta$ uncountable regular cardinals satisfying $\lambda'^{<\lambda'} < \lambda$. Then there is a forcing notion P, $|P| = \delta$, such that if G is generic over P, then in $V[G]$ there is a precipitous ideal on $\mathcal{P}_\kappa\lambda$ whose projection to $\mathcal{P}_\kappa\lambda'$ is not precipitous.*

SKETCH OF PROOF. By Theorem 29 in [FMS], if G is generic over $\mathrm{Lv}(<\delta,\lambda)$, then in $V[G]$, a restriction of the NS ideal on $\mathcal{P}_\kappa\lambda$ is precipitous. They show that NS $\restriction$ IA on $\mathcal{P}_\kappa\lambda$, the restriction of the NS ideal on $\mathcal{P}_\kappa\lambda$ to the set of internally approachable subsets of $\mathcal{P}_\kappa\lambda$, is precipitous (see [FMS], p. 33, for a definition of "internally approachable"). They also show (in the proof of the theorem) that this ideal projects to NS $\restriction$ IA on $\mathcal{P}_\kappa\lambda'$.

Thus, one can first force to make the NS ideal on $\mathcal{P}_\kappa\lambda'$ not precipitous using $P^4_{\kappa,\lambda'}$ (Theorem 3.3). By the remark after the proof of Theorem 3.3, this forcing makes all restrictions of the NS ideal on $\mathcal{P}_\kappa\lambda'$ non-precipitous, as well. In particular, it makes NS $\restriction$ IA (on $\mathcal{P}_\kappa\lambda'$) not-precipitous.

Next, we force to make the supercompact cardinal, δ, be λ^+ by using $\mathrm{Lv}(<\delta,\lambda)$. In this model, by Theorem 29 of [FMS] mentioned above, the NS $\restriction$ IA ideal on $\mathcal{P}_\kappa\lambda$ is precipitous. Further, by Lemma 4.3, the NS $\restriction$ IA ideal on $\mathcal{P}_\kappa\lambda'$ remains non-precipitous in this model, thus giving us an example of a precipitous ideal on $\mathcal{P}_\kappa\lambda$ which projects to a non-precipitous ideal on $\mathcal{P}_\kappa\lambda'$. ∎

In fact, in the model $V[G]$ described above, the NS ideal itself is precipitous (by an unpublished result of the author). Thus we can prove, as in Theorem 4.2 and Theorem 4.5, a direct generalization of Theorem 4.2, namely, we have:

THEOREM 4.6. *Let δ be a supercompact cardinal, $\kappa \leq \lambda' < \lambda < \delta$ uncountable regular cardinals. Assume $\lambda'^{<\lambda'} < \lambda$. Then there is a forcing notion P, $|P| = \delta$, such that if G is a generic filter over P then in $V[G]$ the NS ideal on $\mathcal{P}_\kappa\lambda$ is precipitous, but the NS ideal on $\mathcal{P}_\kappa\lambda'$ is not.*

REMARKS. (1) Theorems 3.3, 4.5 and 4.6 all hold for the notion of "strongly club", as well as for the notion of "club" subsets of $\mathcal{P}_\kappa\lambda$.

(2) By methods similar to those used in [G], one can reduce the hypothesis in Theorems 4.5 and 4.6 from "δ is a supercompact cardinal" to "δ is a Woodin cardinal".

5. Finally, given $\lambda' < \lambda$ regular uncountable cardinals, we describe a model where the NS ideal on $\mathcal{P}_{\omega_1}\lambda$ is presaturated but there are no presaturated ideals on $\mathcal{P}_{\omega_1}\lambda'$:

THEOREM 5.1. *Let δ be a Woodin cardinal, $\lambda' < \lambda < \delta$ uncountable regular cardinals. Assume $\lambda'^{<\lambda'} < \lambda$. Then there is a forcing notion P, $|P| = \delta$, such that if G is a generic filter over P then in $V[G]$ the NS ideal on $\mathcal{P}_{\omega_1}\lambda$ is presaturated, but there are no presaturated ideals on $\mathcal{P}_{\omega_1}\lambda'$.*

PROOF. As before, in the proof of Theorem 4.2, we may assume that $\lambda'^{<\lambda'} = \lambda'$. Let $P^2_{\omega_1,\lambda'}$ be the forcing notion described in section 2 for making all ideals on $P_{\omega_1}\lambda'$ become not presaturated. Let G_1 be generic over $P^2_{\omega_1,\lambda'}$. Then

$$V[G_1] \models \text{there is no presaturated ideal on } \mathcal{P}_{\omega_1}\lambda'.$$

Again, δ is still Woodin in $V[G_1]$, since $|P^2_{\omega_1,\lambda'}| = (\lambda')^+$. Thus if G_2 is $V[G_1]$ generic over $(\mathrm{Lv}(<\delta,\lambda))^{V[G_1]}$ $(=(\mathrm{Lv}(<\delta,\lambda))^V)$ then

$$V[G_1][G_2] \models \text{the NS ideal on } \mathcal{P}_{\omega_1}\lambda \text{ is presaturated.}$$

Thus we will be done if we show that there are still no presaturated ideals on $\mathcal{P}_{\omega_1}\lambda'$ in $V[G_1][G_2]$. This follows from the claim below. Recall the O'_β functions, $O'_\beta : \mathcal{P}_{\omega_1}\lambda \to \omega_1$, that were defined in section 2. Here, as in the previous use of these functions, when we say that a function $f : \mathcal{P}_{\omega_1}\lambda \to \omega_1$ dominates an O'_β function, we mean that there is a club set C_β such that for all X in C_β, $f(X) > O'_\beta(X)$, where C_β is of the form $\{X \in \mathcal{P}_{\omega_1}\lambda \mid \gamma_\beta \in X\}$ for some γ_β in λ.

LEMMA 5.2. *Assume V is a model of ZFC, and f is a function from $\mathcal{P}_{\omega_1}\lambda$ to ω_1 dominating all O'_β functions. Let G be a generic filter over some partial order P in V such that*

$$(\mathcal{P}_{\omega_1}\lambda)^V = (\mathcal{P}_{\omega_1}\lambda)^{V[G]} \textit{ and } \lambda, (\lambda)^+ \textit{ are not collapsed in } V[G].$$

Then in $V[G]$, there are no presaturated (fine) ideals on $\mathcal{P}_{\omega_1}\lambda$.

PROOF. Since $(\mathcal{P}_{\omega_1}\lambda)^V = (\mathcal{P}_{\omega_1}\lambda)^{V[G]}$, the O'_β functions are still functions from $\mathcal{P}_{\omega_1}\lambda$ to λ in $V[G]$. The function O'_β will still represent β in any generic ultrapower by a (fine) normal ideal (and an ordinal $\geq \beta$ in any generic ultrapower by a (fine) ideal). Thus we have functions representing ordinals $< \lambda^+$. Also, f dominates each O'_β function on a set which must belong to each fine filter on $\mathcal{P}_{\omega_1}\lambda$ in $V[G]$. Thus for any fine ideal in $\mathcal{P}_{\omega_1}\lambda$, we have that $j(\omega_1) > \lambda^+$ (where j is a generic embedding associated with the ideal), and thus the ideal is not presaturated. ■

We can now complete the proof that in $V[G_1][G_2]$ there are no presaturated (fine) ideals on $\mathcal{P}_{\omega_1}\lambda'$. The forcing $\mathrm{Lv}(<\delta,\lambda)$ is $<\lambda$-closed and thus $(\mathcal{P}_{\omega_1}\lambda')^V = (\mathcal{P}_{\omega_1}\lambda')^{V[G]}$. Also, no cardinals are collapsed. Thus, by the above lemma, there are no (fine) presaturated ideals on $\mathcal{P}_{\omega_1}\lambda'$. ■

As an immediate corollary we have:

COROLLARY 5.3. *Let $\lambda' < \lambda$ be uncountable regular cardinals, with $\lambda'^{<\lambda'} < \lambda$. Then the existence of a presaturated ideal on $\mathcal{P}_{\omega_1}\lambda$ does not imply the existence of a presaturated ideal on $\mathcal{P}_{\omega_1}\lambda'$. In particular, the existence of a presaturated ideal on $\mathcal{P}_{\omega_1}\lambda$ does not imply the existence of a presaturated ideal on ω_1.*

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